

# A computation in the braid group

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**Abstract.** The braid group is a structure. It has the concepts and the implementations in graphics: points, lines, and some crosses. The crosses are also positive and negative. A braid not only forms graphs, but in different structures based on mathematics. So far, the braids only applies to discrete concepts, but it is possible to present a fuzzy concept. By applying computing to symbols representing the braids, the concept towards fuzzy as generating the form of another structure.

## 1. Introduction

The braids as a system consists of a set of braid that cannot be dynamically changed, but the order can change in such a way as to comprise a pattern of woven forms. Each braid in its set has its own position as a member of the braid [1, 2]. The braids as a product of culture, produces more culture, which can formally be ascertained as a study of mathematics, which involves algebra and geometry. Specifically, the deformation operations meet the requirements of binary operations naturally with their closed, associative, identity and thus every braid has the invers. In case all axioms are met, the braids together with the deformation operation is a braid group [3, 4, 5].

Besides, each braid is formed by a woven pattern:  $\sigma$ ,  $\sigma^{-1}$ , and  $\sigma^0 = 1$  [6], geometrically see Figure 1. However, this paper intends to compute a braid as member of the braid group and prove it through the woven that enable fuzzification process such that the fuzzy braid group is also possible exists.

## 2. Modeling description

Each braid will consist of at least one woven pattern like Figure 1, and will contain many woven. Three woven patterns have different symbols, we specifically write them as  $\sigma^1$ ,  $\sigma^{-1}$ ,  $\sigma^0 = 1$  [8], and recognize as

$\sigma^1$ ,  
 $\sigma^{-1}$ ,  
 $\sigma^0$ ,

in L<sup>A</sup>T<sub>E</sub>X[9]. Writing the strings "sigma", "^" and "1" for  $\sigma^1$  is briefly written as "sigma", while " $\sigma^0$ " is sometimes sufficiently written "1" to act as an identity for the other two "sigma"s. Besides that, the string "sigma", "^", "-1" is the opposite of "sigma", so that sequentially written "sigma" and " $\sigma^{-1}$ " or vice versa, that means it is "1".



Figure 1. Triple of woven patterns

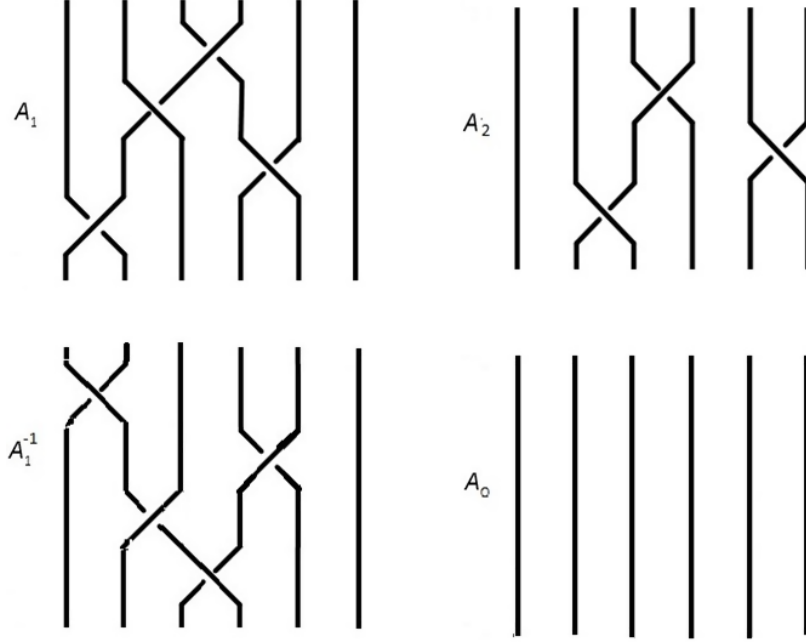


Figure 2. Triple of woven patterns

The woven pattern "sigma^0" forms two parallel lines starting at two point  $i$  and  $i + 1$  and will also be completed at points  $j$  and  $j + 1$ ,  $i = j = 1, \dots, n$ . Whereas the other two patterns, "sigma" and "sigma{-1}" form two lines cross starting at two points  $i$  and  $i + 1$  and ending at the point  $j + 1$  and  $j$ . The number 1 at  $i + 1$  or  $j + 1$  represents the incremental units that give discrete distance to  $i$  or  $j$  respectively. In this case, every line that forms a woven pattern in a braid has the same distance. Therefore, if the woven patterns construct a braid and is composed of lines originating from the point  $i = 1, \dots, n$ , then the lines position the endpoint numbers as a form of permutation. A braid besides having the woven pattern arrangement also has the following permutations [7]:

$$A = \sigma_1^{\pm 1} \sigma_2^{\pm 1} \sigma_3^{\pm 1} \dots \sigma_n^{\pm 1} = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma_1^{\pm 1}(1) & \sigma_2^{\pm 1}(2) & \sigma_3^{\pm 1}(3) & \dots & \sigma_n^{\pm 1}(n) \end{pmatrix} \quad (1)$$

In the formation of braid based on three woven patterns, there are deformation operation that cause the weaving is not the same height to all points of origin, it aims to form the arrangement of woven patterns, "sigma^0", "sigma^1", and "sigma{-1}" does not follow the order of the points  $i = 1, \dots, n$ , but starts from the top woven pattern or whichever comes first. In other words, if  $i$  is the index of each woven pattern, then the arrangement of woven patterns in representing a braid is not follow the order of the index. Based on the lines that have origin and destination points with woven patterns form a permutation. With that provision, the combing operation together with the set of braids forms a group called the braid group with order  $n$ , with the elements, for example, as shown in Figure 2.

For  $A_j$ ,  $j = 1, \dots, m$ , are braids as the members of a braid group  $\mathcal{A}_n$  where  $|A_j|$  is the size of the braid depending on the number of woven that make it, whereas  $|\mathcal{A}|$  is the cardinality of the braid group [6, 10]. Thus, there is a probability

$$p(\sigma) = \frac{|\sigma|}{|A_j|} \quad (2)$$

whereby  $|\sigma|$  is the number of  $\sigma$  on the braid  $A_j$  and  $p(\sigma) \in [0, 1]$ . Otherwise,

$$p(\sigma^{-1}) = 1 - p(\sigma) = \frac{|\sigma^{-1}|}{|A_j|}. \quad (3)$$

whereby  $|\sigma^{-1}|$  is the number of  $\sigma^{-1}$  on the braid  $A_j$ .

A probability by involving Eq. (1) is to compute the probability of exchanging the origin points of the line with the end points of the line, i.e.

$$p(per) = k/n \quad (4)$$

where  $k$  is the number of points in the second line of Eq. (1) it is not the same as the first line of Eq. (1). In this case, the maximum limit is  $n/n = 1$  ( $k = n$ ), and the minimum limit is  $0/n = 0$  ( $k = 0$ ) so  $p(per) \in [0, 1]$ . For example, a braid  $A_1 = \sigma_3^{-1}\sigma_2\sigma_4\sigma_1^{-1}$  by writing it in L<sup>A</sup>T<sub>E</sub>X as follows

`\sigma_3^{-1}\sigma_2\sigma_4\sigma_1^{-1}`

$A_1$  consists of  $|\sigma| = 2$  and  $|\sigma^{-1}| = 2$ , thus  $|A_1| = 4$ . So the probability of  $\sigma$ , Eq. (2), is  $p(\sigma) = \frac{1}{2}$ , while  $p(\sigma^{-1}) = \frac{1}{2}$  based on Eq. (3). Beside, the permutation form of  $A_1$  is

$$A_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 1 & 3 & 6 \end{pmatrix}. \quad (5)$$

Thus, according to Eq. (4), that is  $p(per) = \frac{5}{6}$ .

### 3. Computation

To simplify computing related to Eqs. (2), (3), and (4), for example, and for the purpose of producing an interpretation of the form of a braid, the information technology consists of computer and programming language are the tool for it. By involving the programming language, the interpretation of the braid computationally either involves the woven patterns curves or the string symbol of the woven patterns. The involvement of a computer program to process a string of the woven pattern symbols is to process:

#### 3.1. Normalization

Normalization is to reduce the same or similar symbols in sequence so that there is no duplication. For example,  $\sigma_2\sigma_2$  becomes  $\sigma_2^2$ ,  $\sigma_1^{-1}\sigma_1^{-1}$  changes to  $\sigma_1^{-2}$ ,  $\sigma_3^{-2}\sigma_3$  is abbreviated to  $\sigma_3$ . Normalization aims to reduce the writing of symbols to be short in accordance with the symbol index and to give a new power of the symbol in accordance with the task of the two triples in Figure 1,  $\sigma\sigma^{-1} = \sigma^0 = 1$  [5].

#### 3.2. Declaration

Declaration of the symbol text for Phyton language after going through the normalization process, for example, is to determine the variables in program

```
dt = '\sigma_3^{-1}\sigma_2\sigma_4\sigma_1^{-1}'
pdt = dt.split('\sigma_')
```

A variable `dt` is for storing symbols in the form of text, and a variable `pdt` holds the parts of the symbol as a whole for separate use by computing interests.

### 3.3. Computing about $\sigma$

A computation involves Eqs. (2) and (3), for example, in Python, is

```
for d in pdt:
    if len(d)>0:
        pd = d.split('~')
        if len(pd)>1:
            pd[1]=pd[1].replace('{','')
            pd[1]=pd[1].replace('}','')
            if int(pd[1])>0:
                jsp=jsp+float(pd[1])
            else:
                jsn=jsn+float(pd[1])
        else:
            jsp=jsp+1
print 'p(sigma^{+1}) = ',jsp/(jsp+(-1*jsn))
print 'p(sigma^{-1}) = ',(-1*jsn)/(jsp+(-1*jsn))
```

and the results of process for  $A_1$  are

```
p(sigma^{+1}) = 0.5
p(sigma^{-1}) = 0.5
```

### 3.4. Simulating

Next, in the computational part of the permutation simulation and the result is a probability between the moved and fixed points due to a woven pattern. In the Python programming language, it is as follows

```
print 'first line  = ',brs
for d in pdt:
    pper = 0.0
    per = 0.0
    if len(d)>0:
        pd = d.split('~')
        ss = brd[int(pd[0])-1]
        sd = brd[int(pd[0])]
        brd[int(pd[0])]=ss
        brd[int(pd[0])-1]=sd
        for i in range(len(brs)):
            brn[brd[i]-1]=brs[i]
        for j in range(len(brs)):
            if brs[j]==brn[j]:
                per = per+1
            else:
                pper = pper+1
        print 'move row      = ',brd,'sigma',d
        print 'second line = ',brn
        print 'probability of permutation = ',pper/len(brs)
```

The results of the process for  $A_1$  in the form of moved points and probability are as follows

```
first line  = [1, 2, 3, 4, 5, 6]
move row    = [1, 2, 4, 3, 5, 6] sigma 3^{-1}
```

$p(\sigma^{+1}) = 0.5$ $p(\sigma^{-1}) = 0.5$		
first line	[1, 2, 3, 4, 5, 6]	
move row second line probability of permutation	[1, 2, 4, 3, 5, 6] $\sigma 3^{-1}$ [1, 2, 4, 3, 5, 6] 0.333333333333	[2, 1, 3, 4, 5, 6] $\sigma 1$ [2, 1, 3, 4, 5, 6] 0.333333333333
move row second line probability of permutation	[1, 4, 2, 3, 5, 6] $\sigma 2$ [1, 3, 4, 2, 5, 6] 0.5	[2, 1, 3, 5, 4, 6] $\sigma 4^{-1}$ [2, 1, 3, 5, 4, 6] 0.666666666667
move row second line probability of permutation	[1, 4, 2, 5, 3, 6] $\sigma 4$ [1, 3, 5, 2, 4, 6] 0.666666666667	[2, 3, 1, 5, 4, 6] $\sigma 2^{-1}$ [3, 1, 2, 5, 4, 6] 0.833333333333
move row second line probability of permutation	[4, 1, 2, 5, 3, 6] $\sigma 1^{-1}$ [2, 3, 5, 1, 4, 6] 0.833333333333	[2, 3, 5, 1, 4, 6] $\sigma 3$ [4, 1, 2, 5, 3, 6] 0.833333333333
(a)		
move row second line probability of permutation	[1, 2, 4, 3, 5, 6] $\sigma 3^{-1}$ [1, 2, 4, 3, 5, 6] 0.333333333333	
move row second line probability of permutation	[1, 4, 2, 3, 5, 6] $\sigma 2$ [1, 3, 4, 2, 5, 6] 0.5	[1, 4, 2, 3, 5, 6] $\sigma 4^{-1}$ [1, 3, 4, 2, 5, 6] 0.5
move row second line probability of permutation	[1, 4, 2, 5, 3, 6] $\sigma 4$ [1, 3, 5, 2, 4, 6] 0.666666666667	[1, 2, 4, 3, 5, 6] $\sigma 2^{-1}$ [1, 2, 4, 3, 5, 6] 0.333333333333
move row second line probability of permutation		[1, 2, 3, 4, 5, 6] $\sigma 3$ [1, 2, 3, 4, 5, 6] 0.0
(b)		
move row second line probability of permutation	[1, 2, 4, 3, 5, 6] $\sigma 3^{-1}$ [1, 2, 4, 3, 5, 6] 0.333333333333	
move row second line probability of permutation	[1, 4, 2, 3, 5, 6] $\sigma 2$ [1, 3, 4, 2, 5, 6] 0.5	
move row second line probability of permutation		[1, 2, 4, 3, 5, 6] $\sigma 2^{-1}$ [1, 2, 4, 3, 5, 6] 0.333333333333
move row second line probability of permutation		[1, 2, 3, 4, 5, 6] $\sigma 3$ [1, 2, 3, 4, 5, 6] 0.0
(c)		

**Figure 3.** The steps of reducing  $\sigma$  with using woven patterns

```

second line = [1, 2, 4, 3, 5, 6]
probability of permutation = 0.333333333333
move row    = [1, 4, 2, 3, 5, 6] sigma 2
second line = [1, 3, 4, 2, 5, 6]
probability of permutation = 0.5
move row    = [1, 4, 2, 5, 3, 6] sigma 4
second line = [1, 3, 5, 2, 4, 6]
probability of permutation = 0.666666666667
move row    = [4, 1, 2, 5, 3, 6] sigma 1^{-1}
second line = [2, 3, 5, 1, 4, 6]
probability of permutation = 0.833333333333

```

The first line states the position of the origin represented in the first line of the numbers in Eq. (1) or like Eq. (5). Whereas the second line is to express the moved point in the second line of permutation in Eq. (1) or like Eq. (5). Meanwhile, the move row is a marker of movement of origin at the next position based on woven patterns as the cause.

Another example in computing is the verification of the relationship between a braid  $A_1$  and its invers, a braid  $A_1^{-1}$ , is to produce the identity braid  $A_0 = 1$ ,  $A_1, A_1^{-1}, A_0 \in \mathcal{A}_n$  see Figure 2

[6]. Figure 3 outlines the process that applies to each arrangement and the reduction of symbolic patterns of opposing weaving. Figure 3(a) explains the process towards

$$\sigma_3^{-1}\sigma_2\sigma_4\sigma_1^{-1}\sigma_1\sigma_4^{-1}\sigma_2^{-1}\sigma_3$$

Whereas Figure 3(b) explains the process of

$$\sigma_3^{-1}\sigma_2\sigma_4\sigma_1^{-1}\sigma_2^{-1}\sigma_3$$

After reducing  $\sigma_1^{-1}\sigma_1$ . So by doing the same thing, by reducing  $\sigma_4\sigma_4^{-1}$ ,  $\sigma_2\sigma_2^{-1}$ , etc., it proves that the symbol reduction process follows the triple of woven pattern. Thus the woven patterns is the basic braid also and can as members of a braid group.

#### 4. A review

Based on Artins definition of the braid group [11], the braid theory has become an interesting study for many scientists in mathematics and various applications [12, 13, 14], including applications relating to data security or programs [15, 16, 17]. But so far, there have not been studies that change the braid discrete structure into different interpretations of the structure, as the concept of probability [18], stochastic [19], or fuzzy [10].

In general, an object has different properties. By involving ontology or taxonomic concepts, objects have descriptions based on those properties [20]. The braid as a discrete structure, also has non-discrete properties as an interpretation of the structure attached to it computationally [18, 10]. This depends on the definition [14]. While the evidence of interpretation will strengthen the concept of the presence of a theory related to the braid. So far, for application reasons, computing has to do with the braid also dealing with decomposing the genetic patterns [5, 21]. Although, invariants concept becomes the foundation of braid theory in particular [22], group theory generally, the process of invariants can be implemented as the concept of similarity [23]. For this reason, it is possible to present the concept of a fuzzy braid group, which gives a value to a braid structure in the range of  $[0, 1]$ .

#### 5. Conclusion

Some formulations about the braid computation can be proven for the concept of the fuzzy braid group. A concept that looks at new discrete structures of braid by involving probability, stochastic or fuzzy. However, computational evidence is not enough, without the concept that builds a theory. By involving programming languages, computing the braid group is easier relatively, but abstracting them in theory requires special attention.

#### References

- [1] E Artin 1925 Theorie der Zöpfe *Hamb. Abs.* **4**.
- [2] M K M Nasution 1990 Anyaman: Suatu model matematika *Teori Anyaman* **1(1)**.
- [3] J S Birman 1969 On braid groups *Communications on Pure and Applied Mathematics* **XXII**.
- [4] M K M Nasution 1990 Grup anyaman *Teori Anyaman* **2(1)**.
- [5] M K M Nasution 1992 Studi pembuktian eksistensi simpul normal dari polinomial simpul *Jurusan Matematika, FMIPA USU*.
- [6] M K M Nasution 2018 The braid group: Redefining *MATEC Web of Conferences* **197**.
- [7] C Franchi 2001 On permutation groups of finite type *Europ. J. Combinatorics* **22**.
- [8] M K M Nasution 2000 The number triangle of Laurent polynomial on knots *Jurnal Matematika dan Terapannya* **1(2)**.
- [9] E Herawati, N R G Septian, M O I P Trg 2018 *Panduan Latex untuk Pemula* USU Press.
- [10] M K M Nasution 2018 Fuzzy braid group: A concept *Journal of Physics: Conference Series* **1116**.
- [11] E Artin 1947 Theory of braids *Annals of Mathematics* **48**.
- [12] T E Brendle and D Margalit 2018 The level four braid group *Journal für die reine und angewandte Mathematik* **735**.

- [13] M Baldemor 2019 On mathematical braids *Data Science: Journal of Computing and Applied Informatics (JoCAI)* **3(2)**.
- [14] Nasution M K M, Harahap A S, Nasution S, and Herawati E 2018 Triangle number of Laurent polynomial for the closed braids  $\sigma^{2j}$  *IOP Conf. Series: Materials Science and Engineering* **300**.
- [15] J Hughes 2002 A linear algebra attack on the AAFG1 braid group cryptosystem *Lecture Notes in Computer Science* **2384**.
- [16] Gebhardt V 2006 Conjugacy search in braid groups from a braid-based cryptography point of view *AAECC* **17**.
- [17] G T Simbolon, O S Sitompul, E B Nababan 2019 Data Security Using Multi-bit LSB and Modified Vernam Cipher *Data Science: Journal of Computing and Applied Informatics (JoCAI)* **3(1)**.
- [18] D Garber, S Kaplan, M Teicher, B Tsaban, U Vishne 2005 Probabilistic solutions of equations in the braid group *Advances in Applied Mathematics* **35**.
- [19] J Ren, N A Sinitsyn 2013 Braid group and topological phase transitions in nonequilibrium stochastic dynamics *Phys. Rev. E* **87**.
- [20] M K M Nasution 2018 Ontology *Journal of Physics: Conference Series*.
- [21] T Charvin, A Vologodskii, D Bensimon, V Croquette 2005 Braiding DNA: Experiments, simulations, and models *Biophysical Journal* **88(6)**.
- [22] P J Olver, P Rosenau 2006 Group-invariant solutions of differential equations *SIAM J. Appl. Math.* **47(2)**.
- [23] M K M Nasution, O S Sitompul, S Nasution, H Ambarita 2017 New Similarity *IOP Conference Series: Materials Science and Engineering* **180(1)**.